Elementary first-order model checking for sparse graphs

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Model checking first-order formulas (on graphs)

first-order logic (**FO**):

Atomic formulas: \( x = y, \ adj(x, y) \)
Logical connectives: \( \varphi \land \psi, \ \varphi \lor \psi, \ \neg \varphi \).
Quantifiers: \( \exists x \ \varphi, \ \forall x \ \varphi \)
Model checking first-order formulas (on graphs)

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Logical connectives: $\varphi \land \psi$, $\varphi \lor \psi$, $\neg \varphi$
Quantifiers: $\exists x \varphi$, $\forall x \varphi$

"$P_3$ is an induced subgraph of $G$":

$\exists x \exists y \exists z \left( \text{adj}(x, y) \land \text{adj}(y, z) \land \neg \text{adj}(x, z) \right)$

"$G$ has a dominating set of size 3":

$\exists x_1 \exists x_2 \exists x_3 \forall y \bigvee_{i \in \{1, 2, 3\}} \left( y = x_i \lor \text{adj}(x_i, y) \right)$
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FO Model Checking

Input: a first-order formula $\varphi$ and a graph $G$.

Question: $G$ satisfies $\varphi$?
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FO Model Checking
Input: a first-order formula $\varphi$ and a graph $G$. Question: $G$ satisfies $\varphi$?

- On general graphs, the problem is AW[*]-hard.
- When is it FPT? i.e., solvable in time $f(|\varphi|, C) \cdot |G|^c$, for some function $f$ and $c \geq 1$. 

\[ \]
The three components of the model checking question

**FO** model checking is **FPT** on $C$. 
The three components of the model checking question

[Dreier, Eleftheriadis, Mählmann, McCarty, Pilipczuk, & Toruńczyk, 2023]
[Dreier, Mählmann, & Siebertz, 2023]
[Bonnet, Dreier, Gajarský, Kreutzer, Mählmann, Simon, & Toruńczyk, 2022]
[Bonnet, Kim, Thomassé, & Watrigant, 2022]
[Hliněný, Pokrývka, & Roy, 2019]
[Gajarský, Kreutzer, Nešetřil, Ossona de Mendez, Pilipczuk, Siebertz, & Toruńczyk, 2018]
[Grohe, Kreutzer, & Siebertz, 2017]
[Eickmeyer & Kawarabayashi, 2017]
[Gajarský, Hliněný, Lokshtanov, Obdržálek, & Ramanujan, 2016]
[Dvořák, Král, & Thomas, 2011]
[Dawar, Grohe, & Kreutzer, 2007]
[Flum & Grohe, 2001]
[Frick & Grohe, 2001]
[Seese, 1996]

FO model checking is FPT on $C$.

How general $C$ can be?
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[Dreier, Eleftheriadis, Mählmann, McCarty, Pilipczuk, & Toruńczyk, 2023]
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Extensions of \( \text{FO} \) ? \( \text{FO} \) model checking is \( \text{FPT} \) on \( C \). How general \( C \) can be?

[Schirrmacher, Siebertz, Stamoulis, Thilikos, & Vigny, 2023]
[Golovach, Stamoulis, & Thilikos, 2023]
[Fomin, Golovach, Sau, Stamoulis, & Thilikos, 2023]
[Pilipczuk, Schirrmacher, Siebertz, Toruńczyk, & Vigny, 2022]
[Schirrmacher, Siebertz, & Vigny, 2022]
[Nešetřil, Ossona de Mendez, & Siebertz, 2022]
[Grange, 2021]
[Berkholz, Keppeler, & Schweikardt, 2018]
[Grohe & Schweikardt, 2018]
[van den Heuvel, Kreutzer, Pilipczuk, Quiroz, Rabinovich, & Siebertz, 2017]
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[Dreier, Eleftheriadis, Mählmann, McCarty, Pilipczuk, & Toruńczyk, 2023]
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Extensions of \( \text{FO} \)?

\( \text{FO} \) model checking is \( \text{FPT} \) on \( C \).

How general \( C \) can be?

What about “elementarily-\( \text{FPT} \)”?
What is the (parametric) dependence on $|\phi|$ in the running time of a model checking algorithm?

Even for the class $T$ of trees. [Frick & Grohe, 2002]

Task: 
Improve the (parametric) dependence on $|\phi|$ in the running time.

Input: a first-order formula $\phi$ and a graph $G \in C$

Question: $G$ satisfies $\phi$?

Meta-parameter:
Elementarily-FPT: running time $2^{2^n \cdots^{2^n}} |\phi| |\{z\}| \cdot |G| c^3 / 20$
“Elementarily-FPT” programme

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$$2^{2^{|\varphi|}} \cdot |G|^c, \text{ for some constant } c \geq 1,$$

$\text{height } g(|\varphi|)$
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for some constant $c \geq 1$, even for the class $\mathcal{T}$ of trees. [Frick & Grohe, 2002]
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Task: Improve the (parametric) dependence on $|\varphi|$ in the running time.

**FO Model Checking (on $\mathcal{C}$)**
Input: a first-order formula $\varphi$ and a graph $G \in \mathcal{C}$
Question: $G$ satisfies $\varphi$?
“Elementarily-FPT” programme

What is the (parametric) dependence on $|\varphi|$ in the running time of a model checking algorithm?

$$2^{2^{2|\varphi|}} \cdot |G|^c,$$

for some constant $c \geq 1$, even for the class $\mathcal{T}$ of trees. [Frick & Grohe, 2002]

**Task:** Improve the (parametric) dependence on $|\varphi|$ in the running time.

**FO Model Checking (on $\mathcal{C}$)**
Input: a first-order formula $\varphi$ and a graph $G \in \mathcal{C}$
Question: $G$ satisfies $\varphi$?

**Meta-parameter:** $h_C$
“Elementarily-FPT” programme

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**FO Model Checking (on $\mathcal{C}$)**

- **Input:** a first-order formula $\varphi$ and a graph $G \in \mathcal{C}$
- **Question:** $G$ satisfies $\varphi$?

**Elementarily-FPT:** running time

$$2^{2^{|\varphi|}} \cdot |G|^c \cdot \text{height } g(h_{\mathcal{C}})$$
**elementary function**: can be formed from

- successor function
- addition/subtraction/multiplication

using

* compositions,
* projections,
* bounded additions/multiplications.
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**Observation**: 

A function \( f \) is bounded by an elementary function \( \iff \) it is bounded by an \( h \)-fold exponential function for some fixed \( h \).
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a function $f$ is bounded by an elementary function
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**Elementarily-FPT**: running time 
\[
2^{2^{\varphi}} \cdot |G|^c \cdot \text{height } g(h_c)
\]
The map of the elementarily-FPT universe

- **Bounded pathwidth**
  [Lampis, 2023]

- **Bounded treedepth**
  [Gajarský & Hlinený, 2015]

- **Bounded degree**
  [Frick & Grohe, 2002]
The map of the elementarily-FPT universe

- Bounded pathwidth
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- ?
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What **bounded degree** and **bounded pathwidth** have **in common**?
What \textit{bounded degree} and \textit{bounded pathwidth} have \textit{in common}?

\textbf{Exclusion of a tree as a topological minor}
What bounded degree and bounded pathwidth have in common?

Exclusion of a tree as a topological minor

*Elementary model checking for classes excluding a tree $T$ as a topological minor?*
What bounded degree and bounded pathwidth have in common?

Exclusion of a tree as a topological minor

*Elementary* model checking for classes *excluding a tree* $T$ *as a topological minor*?

*If yes, how more general* can we get?
Definitions:

\[ H(\leq r) := \text{replace every edge of } H \text{ with a path of at most } r \text{ internal vertices.} \]

\[ H \text{ is an } r\text{-shallow topological minor of } G, \text{ if } H(\leq r) \subseteq G. \]

\[ \text{TopMinors}_r(C) := \{ H | \exists G \in C: H \text{ is an } r\text{-shallow topological minor of } G \} \]

\[ T_d := \text{class of all trees of depth } d. \]

\[ \bullet \text{The tree rank of a graph class } C := \max \{ d \in \mathbb{N} | \exists r \in \mathbb{N}: T_d \subseteq \text{TopMinors}_r(C) \}. \]
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- The tree rank of a graph class $C$: $\max\{d \in \mathbb{N} \mid \exists r \in \mathbb{N} : T_d \subseteq \text{TopMinors}_r(C)\}$.
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What is bounded tree rank?

- The class $\mathcal{T}$ of all trees has unbounded tree rank.
What is bounded tree rank?

- The class $\mathcal{T}$ of all trees has \textit{unbounded} tree rank.
- $\mathcal{T}_d$ has tree rank $d$. 
What is bounded tree rank?

- The class $\mathcal{T}$ of all trees has unbounded tree rank.
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- If $\mathcal{C}$ excludes some tree $T$ as a topological minor, it has tree rank smaller than the depth of $T$. 
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- $\mathcal{C}$ has bounded degree if and only if $\mathcal{C}$ has tree rank 1.
- The class $\mathcal{C}$ of graphs of pathwidth $d$ has tree rank exactly $d + 1$. 
Is this just excluding a tree as a topological minor?
Is this just excluding a tree as a topological minor? **NO**
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Every tree as a topological minor and tree rank 2
Fact: A graph of minimum degree $\delta$ contains every tree on $\delta$ vertices as a subgraph.

$\text{bounded tree rank} \Rightarrow \text{bounded degeneracy} \Rightarrow \text{bounded expansion}$
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bounded expansion $\Rightarrow$ bounded tree rank $\Rightarrow$ nowhere dense

excluding a tree as a topological minor

bounded tree rank =

$\Rightarrow$

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bounded tree rank $\implies$ bounded degeneracy $\implies$ bounded expansion
$T^d_k :=$ tree of depth $d$ and branching/size $k$.

**Tree rank of $C$:**
The least number $d \in \mathbb{N}$ such that for every $r \in \mathbb{N}$ there is $k \in \mathbb{N}$ s.t. no graph in $C$ contains $T^{d+1}_k$ as an $r$-shallow topological minor.
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**Tree rank of $C$:**
the least number $d \in \mathbb{N}$ such that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that
for every $r \in \mathbb{N}$, no graph in $C$ contains $T_{f(r)}^{d+1}$ as an $r$-shallow topological minor.
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**Elementary tree rank of \( C \):**
the least number \( d \in \mathbb{N} \) such that there is an elementary function \( f : \mathbb{N} \to \mathbb{N} \) such that
for every \( r \in \mathbb{N} \), no graph in \( C \) contains \( T^{d+1}_{f(r)} \) as an \( r \)-shallow topological minor.
Elementary FO model checking on sparse classes

Theorem [Gajarský, Pilipczuk, Sokołowski, Stamoulis, Toruńczyk, 2023]

If $C$ has bounded elementary tree rank, then FO model checking is elementarily-FPT on $C$.

Corollary

If $C$ excludes a fixed tree as a topological minor, then FO model checking is elementarily-FPT on $C$.

Theorem [Gajarský, Pilipczuk, Sokołowski, Stamoulis, Toruńczyk, 2023]

Assume AW[*]̸=FPT. Let $C$ be a monotone graph class. If FO model checking is elementarily-FPT on $C$, then $C$ has bounded tree rank.

Almost complete characterization of elementarily-FPT FO model checking on sparse classes.
Elementary FO model checking on sparse classes

**Theorem** [Gajarský, Pilipczuk, Sokołowski, Stamoulis, Toruńczyk, 2023]

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Elementary FO model checking on sparse classes

Theorem [Gajarský, Pilipczuk, Sokołowski, Stamoulis, Toruńczyk, 2023]
If $\mathcal{C}$ has bounded elementary tree rank, then FO model checking is elementarily-FPT on $\mathcal{C}$.

Corollary
If $\mathcal{C}$ excludes a fixed tree as a topological minor, then FO model checking is elementarily-FPT on $\mathcal{C}$.

Theorem [Gajarský, Pilipczuk, Sokołowski, Stamoulis, Toruńczyk, 2023]
Assume $\text{AW}[\ast] \neq \text{FPT}$. Let $\mathcal{C}$ be a monotone graph class.
If FO model checking is elementarily-FPT on $\mathcal{C}$, then $\mathcal{C}$ has bounded tree rank.
Elementary FO model checking on sparse classes

**Theorem** [Gajarský, Pilipczuk, Sokołowski, Stamoulis, Toruńczyk, 2023]
If \( C \) has bounded elementary tree rank, then FO model checking is elementarily-FPT on \( C \).

**Corollary**
If \( C \) excludes a fixed tree as a topological minor, then FO model checking is elementarily-FPT on \( C \).

**Theorem** [Gajarský, Pilipczuk, Sokołowski, Stamoulis, Toruńczyk, 2023]
Assume AW[*] ≠ FPT. Let \( C \) be a monotone graph class. If FO model checking is elementarily-FPT on \( C \), then \( C \) has bounded tree rank.

*Almost* complete characterization of elementarily-FPT FO model checking on sparse classes.
Collapse of FO alternation hierarchy

Lemma

Let \( C \) be a graph class of tree rank \( d \).
Every formula \( \phi \) is equivalent on \( C \) to a formula \( \psi \) of alternation rank \( 3d \).

Also, if \( C \) has elementary tree rank \( d \), then \( |\psi| \) is elementary in \( |\phi| \).

\( \exists \forall \exists \exists \forall \exists \forall \forall \forall \).

Theorem

[Gajarský, Pilipczuk, Sokołowski, Stamoulis, Toruńczyk, 2023]

Let \( C \) be a monotone graph class. The following are equivalent:

\( C \) has bounded tree rank

\( \exists k \in \mathbb{N} \) such that for every formula \( \phi \), there is an equivalent (on \( C \)) formula \( \psi \) of alternation rank \( k \).
Collapse of FO alternation hierarchy

Lemma

Let $C$ be a graph class of tree rank $d$. Every formula $\varphi$ is equivalent on $C$ to a formula $\psi$ of alternation rank $3d$.

Also, if $C$ has elementary tree rank $d$, then $|\psi|$ is elementary in $|\varphi|$. 

Theorem [Gajarský, Pilipczuk, Sokołowski, Stamoulis, Toruńczyk, 2023]

Let $C$ be a monotone graph class. The following are equivalent:

- $C$ has bounded tree rank $\exists k \in \mathbb{N}$ such that for every formula $\varphi$, there is an equivalent (on $C$) formula $\psi$ of alternation rank $k$.
Lemma

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**Theorem** [Gajarský, Pilipczuk, Sokołowski, Stamoulis, Toruńczyk, 2023]

Let $C$ be a monotone graph class. The following are equivalent:

- $C$ has bounded tree rank
- $\exists k \in \mathbb{N}$ such that for every formula $\varphi$, there is an equivalent (on $C$) formula $\psi$ of alternation rank $k$. 
Structural characterization of bounded tree rank

$m$-batched splitter game of radius $r$:
Structural characterization of bounded tree rank

$m$-batched splitter game of radius $r$:

Two players: \textbf{Splitter} and \textbf{Localizer}. 
Structural characterization of bounded tree rank

*m-batched splitter game of radius* $r$:

Two players: **Splitter** and **Localizer**. In each round of the game:
Structural characterization of bounded tree rank

$m$-batched splitter game of radius $r$:

Two players: **Splitter** and **Localizer**. In each round of the game:

- **Localizer** picks $v \in V(G)$ and restricts to $G' := B'_G(v)$.

Lemma

Let $d \in \mathbb{N}$. The following conditions are equivalent:

1. $C$ has (elementary) tree rank $d$,
2. There is an (elementary) function $f: \mathbb{N} \to \mathbb{N}$ such that for every $r \in \mathbb{N}$ Splitter wins the $f(r)$-batched splitter game of radius $r$ in at most $d$ rounds, on every $G \in C$. 

$G$
Structural characterization of bounded tree rank

$m$-batched splitter game of radius $r$:

Two players: **Splitter** and **Localizer**. In each round of the game:

- **Localizer** picks $v \in V(G)$ and restricts to $G' := B_r^G(v)$.

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G

\[ G \]

\[ v \]

\[ r \]
Structural characterization of bounded tree rank

$m$-batched splitter game of radius $r$: 

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$G'$
Structural characterization of bounded tree rank

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- **Localizer** picks $v \in V(G)$ and restricts to $G' := B'_G(v)$.
- **Splitter** deletes at most $m$ vertices from $G'$ and the game continues on the obtained graph.
Structural characterization of bounded tree rank

m-batched splitter game of radius r:

Two players: **Splitter** and **Localizer**. In each round of the game:

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\textit{m-batched splitter game of radius }r:\n
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Structural characterization of bounded tree rank

\textit{m-batched splitter game of radius} \(r\):

Two players: \textbf{Splitter} and \textbf{Localizer}. In each round of the game:

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\textbf{Lemma}

Let \(d \in \mathbb{N}\). The following conditions are equivalent:

1. \(\mathcal{C}\) has (elementary) tree rank \(d\),
Structural characterization of bounded tree rank

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Lemma

Let \( d \in \mathbb{N} \). The following conditions are equivalent:

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Proof sketch:

\( (2) \Rightarrow (1) \):

How can the Localiser survive \( d + 1 \) rounds in \( (T_d + f(r)) \leq r \) (for radius \( d(r + 1) \))?

\( (1) \Rightarrow (2) \):

Let \( v \in V(G) \) and \( S \subseteq B(\leq r) G(v) \).

If \( |S| > tr \), there is an \( (\leq r) \)-subdivided \( t \)-star in \( B(\leq r) G(v) \) with root \( v \) and leaves \( s_1, \ldots, s_t \in S \).

\( \{ \text{"candidate roots" for } T_i \text{ as an } r \times \text{shallow topological minor in } B(\leq r) G(v) \} \leq f(d, r, k) \)
Lemma

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$\triangleright$
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How to prove the alternation collapse?

Collapse of FO alternation hierarchy

Let $C$ be a graph class of tree rank $d$. Every formula $\varphi$ is equivalent on $C$ to a formula $\psi$ of alternation rank $3d$. 

Proof sketch:

• Gaifman’s Locality Theorem.
  "for every FO-formula $\varphi$, there is a radius $r$ such that the satisfaction of $\varphi$ in $G$ only depends on FO-definable properties of radius-$r$ balls in $G".

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How to do elementary FO model checking?

Compute the "constant alternation rank"-type of the graph, using FO model checking algorithm on bounded expansion classes (which is elementarily-FPT for sentences of constant alternation rank).

The collapse of the FO alternation hierarchy on bounded tree rank classes implies the following:

If two vertices have the same "constant alternation rank"-type, then they have the same $q$-type.
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Conclusion

**Theorem [Gajarský, Pilipczuk, Sokołowski, Stamoulis, Toruńczyk, 2023]**
If $C$ has **bounded elementary tree rank**, then FO model checking is **elementarily-FPT** on $C$.

**Corollary**
If $C$ excludes a fixed tree as a topological minor, then FO model checking is **elementarily-FPT** on $C$.

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**Theorem [Gajarský, Pilipczuk, Sokołowski, Stamoulis, Toruńczyk, 2023]**
Assume $\text{AW}[\ast]\neq \text{FPT}$. Let $C$ be a monotone graph class.
If FO model checking is **elementarily-FPT** on $C$, then $C$ has **bounded tree rank**.

*Almost* complete characterization of **elementarily-FPT** FO model checking on sparse classes.
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Almost complete characterization of elementarily-FPT FO model checking on sparse classes.

What about dense classes?
Towards dense graph classes

Tree rank of $C$: the largest number $d \in \mathbb{N}$ such that there is an $r \in \mathbb{N}$ such that $T_d \subseteq \text{TopMinors}_r(C)$.

Rank of $C$: the largest number $d \in \mathbb{N}$ such that $C$ transduces $T_d$.

Conjecture: A hereditary graph class $C$ has elementarily-FPT model checking if and only if it has bounded rank.

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A graph class $C$ is **weakly sparse** if it excludes some biclique as a subgraph.
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**Theorem** [Gajarský, Pilipczuk, Sokołowski, Stamoulis, Toruńczyk, 2023]

Let $\mathcal{C}$ be a weakly sparse graph class. $\mathcal{C}$ has bounded **tree rank** $\iff$ $\mathcal{C}$ has bounded **rank**.
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Merci!