Elementary first-order model checking for sparse graphs

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Model checking first-order formulas (on graphs)

first-order logic (**FO**):

Atomic formulas: $x = y$, $adj(x, y)$
Logical connectives: $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$.
Quantifiers: $\exists x\ \varphi$, $\forall x\ \varphi$

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- When is it **FPT**? i.e., solvable in time $f(|\varphi|, \mathcal{C}) \cdot |G|^c$, for some function f and $c \geq 1$.

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*What about “elementarily-**FPT**”?*

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- successor function
- addition/subtraction/multiplication

using

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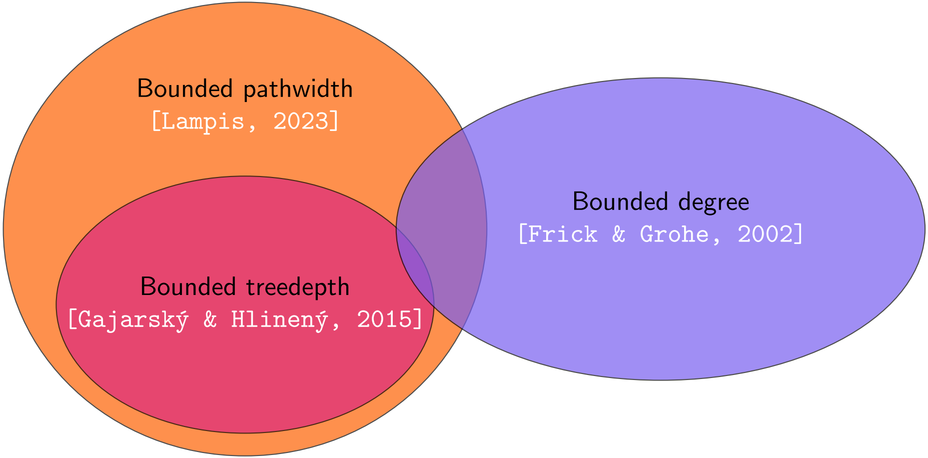
The map of the elementarily-FPT universe



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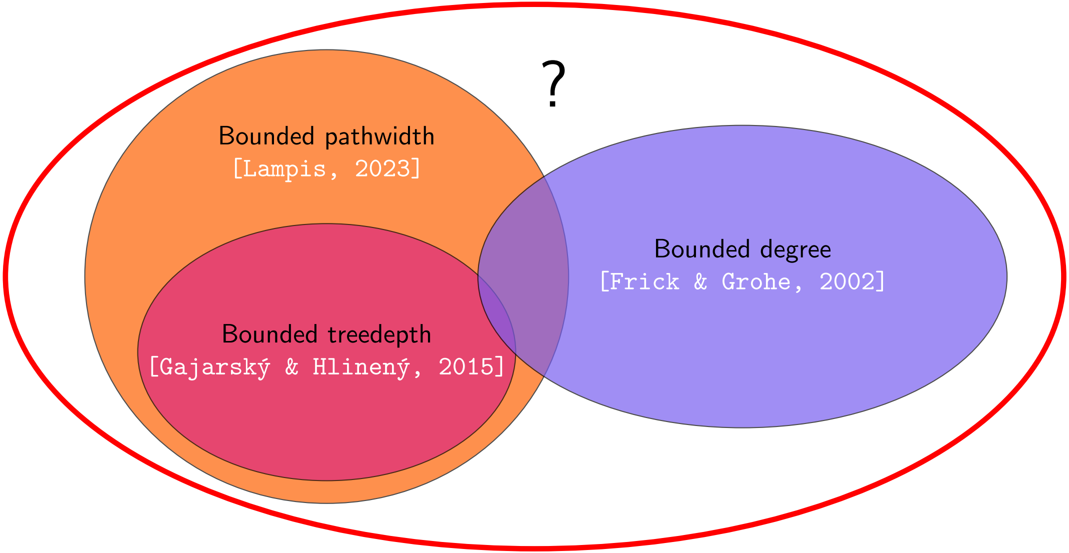

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Elementary model checking for classes excluding a tree T as a topological minor?

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If yes, how **more general** *can we get?*

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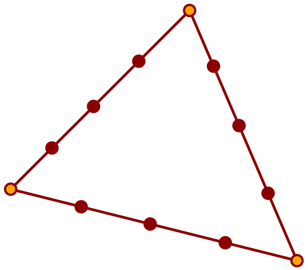

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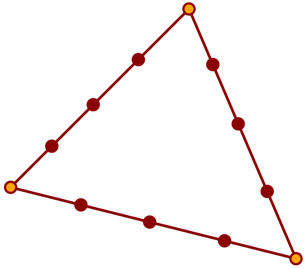

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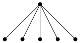
- $\mathcal{T}_d$:= class of all trees of depth d. ( has depth 2.)

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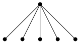
- $\mathcal{T}_d$:= class of all trees of depth d. ( has depth 2.)
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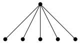
- $\mathcal{T}_d$:= class of all trees of depth d. ( has depth 2.)
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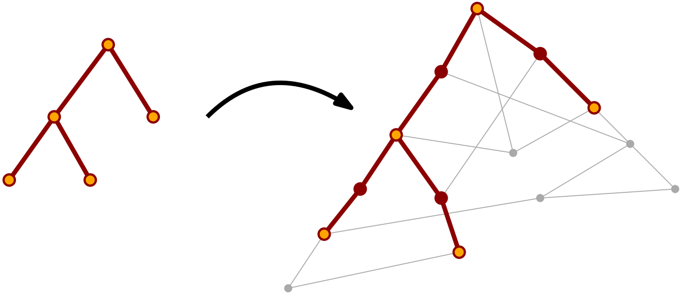

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- The class $\mathcal{C}$ of graphs of pathwidth d has tree rank exactly $d + 1$.

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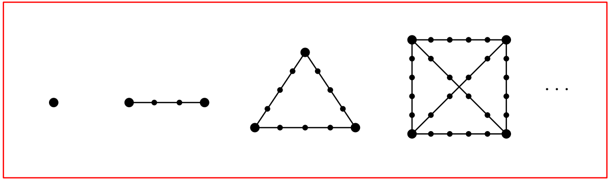

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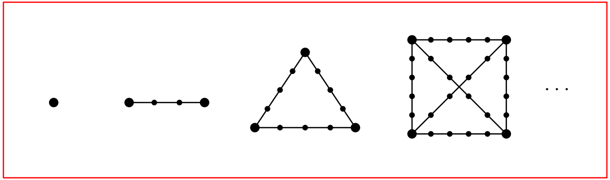

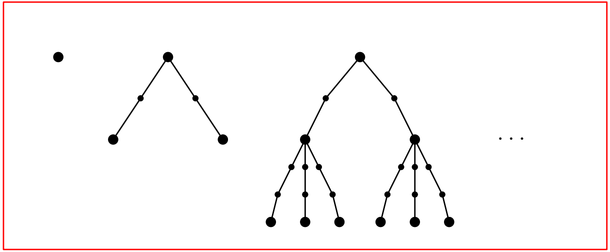

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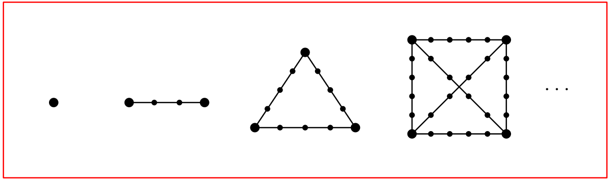
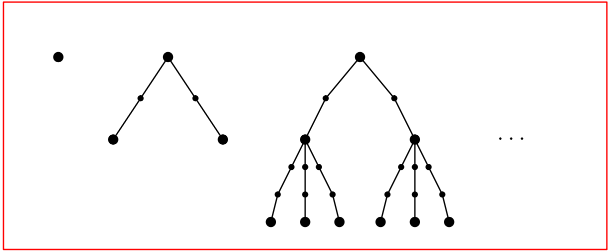
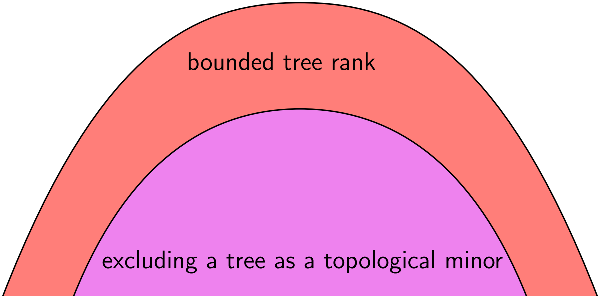




Every tree as a topological minor and tree rank 2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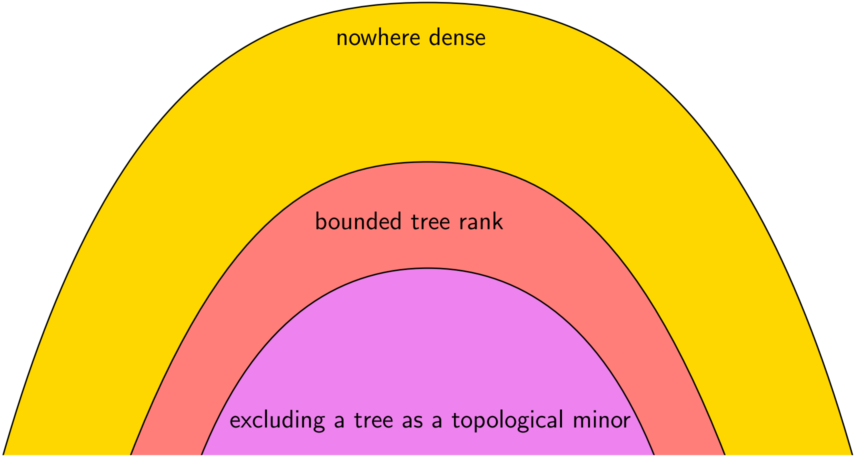
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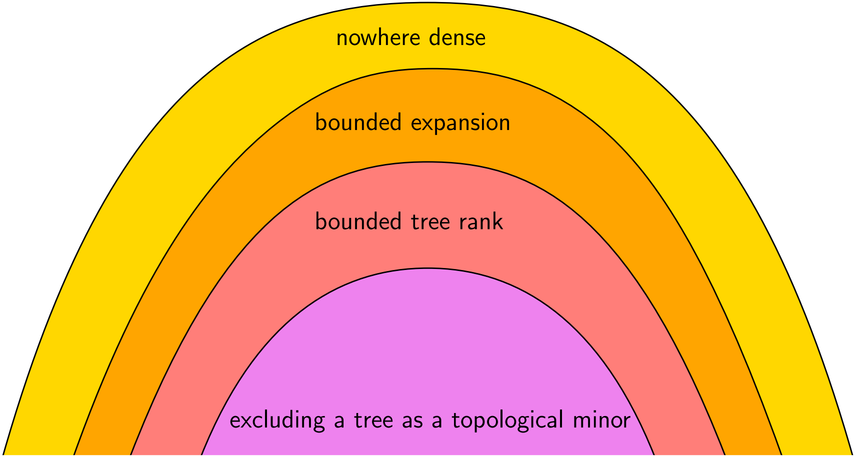
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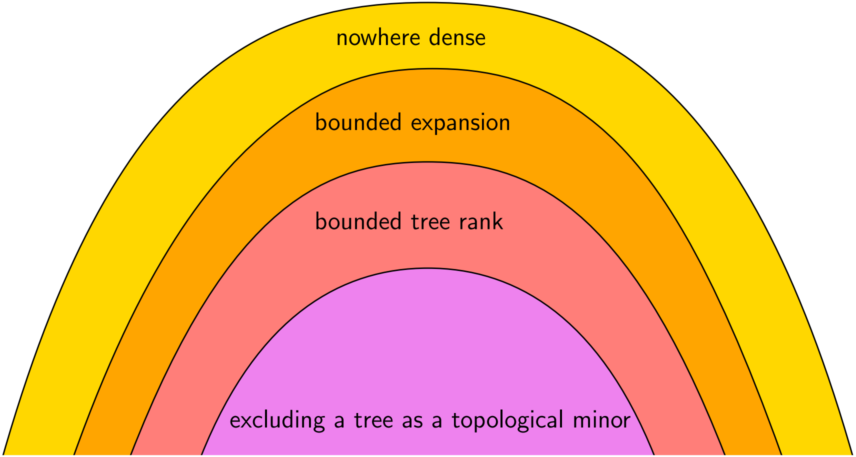

Fact: A graph of minimum degree δ contains every tree on δ vertices as a subgraph.

bounded tree rank $\Longrightarrow$ bounded degeneracy $\Longrightarrow$ bounded expansion

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If $\mathcal{C}$ has bounded elementary tree rank, then FO model checking is elementarily-FPT on $\mathcal{C}$.

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Almost complete characterization of elementarily-FPT FO model checking on sparse classes.

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Let $\mathcal{C}$ be a graph class of tree rank d.
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Theorem [Gajarský, Pilipczuk, Sokołowski, *Stamoulis*, Toruńczyk, 2023]

Let $\mathcal{C}$ be a <u>monotone</u> graph class. The following are equivalent:

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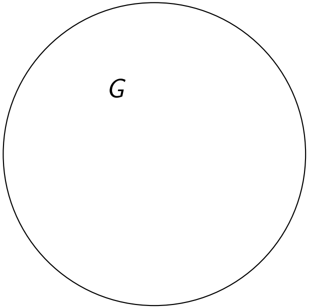

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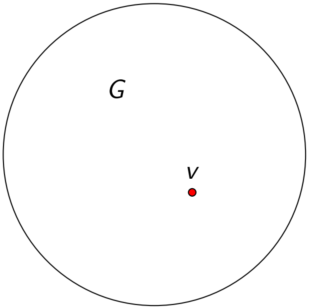

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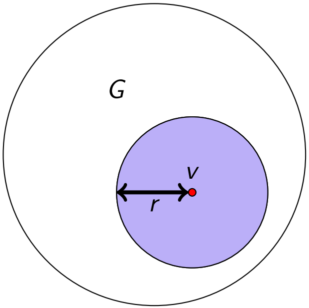

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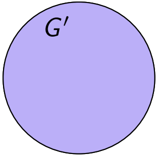

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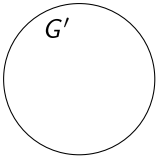

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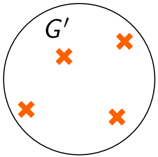

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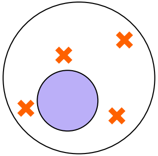

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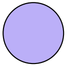

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▷ $\left|\{\text{“candidate roots” for } T_{k'}^i \text{ as an } r\text{-shallow topological minor in } B_G^{(\leq r)}(v)\}\right| \leq f(d, r, k)$

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- Apply induction on every radius-r ball in G, after removing $f(r)$ vertices.

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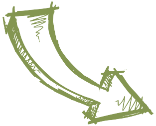

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The collapse of the FO alternation hierarchy on bounded tree rank classes implies the following:

If two vertices have the same "constant alternation rank"-type, then they have the same q*-type.*

Conclusion

Theorem [Gajarský, Pilipczuk, Sokołowski, *Stamoulis*, Toruńczyk, 2023]

If $\mathcal{C}$ has bounded elementary tree rank, then FO model checking is elementarily-FPT on $\mathcal{C}$.

Corollary

If $\mathcal{C}$ excludes a fixed tree as a topological minor, then FO model checking is elementarily-FPT on $\mathcal{C}$.

Theorem [Gajarský, Pilipczuk, Sokołowski, *Stamoulis*, Toruńczyk, 2023]

Assume AW[*]$\neq$FPT. Let $\mathcal{C}$ be a monotone graph class.
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Merci!